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On the maximum value of Jacobi polynomials

Ilia Krasikov

Department of Mathematical Sciences, Brunel University, Uxbridge UB8 3PH, UK

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Abstract

A remarkable inequality, with utterly explicit constants, established by Erdélyi, Magnus, and Nevai, states that for $\alpha \geqslant \beta > -\frac{1}{2}$, the orthonormal Jacobi polynomials $\mathbf{P}_k^{(\alpha,\beta)}(x)$ satisfy

$$\max_{|x|\leqslant 1} \left\{ (1-x)^{\alpha+1/2}(1+x)^{\beta+1/2}\left(\mathbf{P}_k^{(\alpha,\beta)}(x)\right)^2 \right\} = O(\alpha)$$

[Erdélyi et al., Generalized Jacobi weights, Christoffel functions, and Jacobi polynomials, SIAM J. Math. Anal. 25 (1994), 602–614]. They conjectured that the real order of the maximum is $O(\alpha^{1/2})$. Here we will make half a way towards this conjecture by proving a new inequality which improves their result by a factor of order $(\frac{1}{\alpha}+\frac{1}{k})^{-1/3}$. We also confirm the conjecture, even in a stronger form, in some limiting cases.

1. Introduction

In this paper we will use bold letters for orthonormal polynomials versus regular characters for orthogonal polynomials in the standard normalization [6,20].

Let $\mathbf{p}_k$ be an orthonormal polynomial $\mathbf{p}_k$ of degree k, orthogonal with respect to a nonnegative weight function W, supported on finite or infinite interval I (we deal exceptionally with the classical case). Let also $\phi(x)$ be a given auxiliary function nonnegative on I. In this paper $\phi(x)$ will be chosen depending only on a specific family of polynomials in the Askey

E-mail address: mastiik@brunel.ac.uk.

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scheme, e.g we set $\phi = \sqrt{1-x^2}$ and $\sqrt{x}$, for all Jacobi and all Laguerre polynomials respectively. We define the following functions

$$M(x; \mathbf{p}_k, \phi) = W(x)\phi(x)\mathbf{p}_k^2(x), \tag{1}$$

and the functional

$$\mathcal{M}(\mathbf{p}_k, \phi) = \max_{x\in I} \left\{ M(x; \mathbf{p}_k, \phi) \right\}. \tag{2}$$

Thus, for the Jacobi case we have

$$\mathcal{M}\left(\mathbf{P}_k^{(\alpha,\beta)}, \sqrt{1-x^2} \right) = \max_{-1\leqslant x\leqslant 1} \left\{ (1-x)^{\alpha+1/2}(1+x)^{\beta+1/2} \left(\mathbf{P}_k^{(\alpha,\beta)}(x) \right)^2 \right\}.$$

The origin of the function ϕ is, of course, the Szegö theory and its extensions stating that for a wide class of polynomials orthonormal on $[-1, 1]$ and $k \to \infty$, the expression $(1-x^2)^{1/4}\, W^{1/2}\, \mathbf{p}_k$ mimics in a sense the behaviour of the Chebyshev polynomials T_k, equioscillating between $\pm\sqrt{2/\pi}$, [17,20]. Similar results probably hold for many other families, including classical discrete polynomials. The deepest statement in this direction asserts that for exponential weights $W = e^{-Q}$, under some not too restrictive, but rather technical conditions,

$$\max_I \left\{ W \, |(x-a_{-k})(x-a_k)|^{1/2} \, \mathbf{p}_k^2(x) \right\} < C, \tag{3}$$

with C independent on k, and $a_{\pm k}$ being the Mhaskar–Rahmanov–Saff numbers for Q [13–15]. Yet the real reason for such a behaviour is probably hidden not in weights but rather in much poorly understood properties of the coefficients of the three term recurrence.

Whereas many inequalities on $\mathcal{M}$ are known for classical orthogonal polynomials for properly restricted parameters (see e.g. [1,2,5] and the references therein), the main problem we are trying to deal with is to estimate $\mathcal{M}$ uniformly for the entire family. That is, bringing the Jacobi case as an example, to supply tight bounds uniform in k, α and β. An astonishing fact is that this is sometimes possible under marginal restrictions on the parameters. Yet, the only known examples, putting aside asymptotics, are the Hermite polynomials with just one parameter k involved [9] (see (12) below), and the following remarkable inequality, established by Erdélyi et al. [2],

Theorem 1.

$$\mathcal{M}\left(\mathbf{P}_k^{(\alpha,\beta)}, \sqrt{1-x^2} \right) \leqslant \frac{2e\left(2+\sqrt{\alpha^2+\beta^2}\right)}{\pi}, \tag{4}$$

provided $k \geqslant 0$, *and* $\alpha, \ \beta > -\frac{1}{2}$.

A surprising independence on k in (4) is probably an artefact and restored in sharper bounds by the customary multiplier $k^{-1/6}$. They also conjectured that (4) can be tighten to $\left(\alpha^2+\beta^2\right)^{1/4}$.

In this paper we will improve (4) and provide some evidences that the real order of $\mathcal{M}\left(\mathbf{P}_k^{(\alpha,\beta)}, \sqrt{1-x^2}\right)$ is $\sqrt{\alpha}\, k^{-1/6}$, with $\alpha \geqslant \beta$. We will use a method suggested in [7], which, at least in principal, can be adapted for orthogonal polynomials with known second order differential or difference equations. Moreover, it seems that the only real obstacle for getting explicit asymptotically sharp estimates in most of the cases is extremely messy calculations needed to bound zeroes of multivariable polynomials. As a matter of fact, our proof is quite similar to that of [2]. We just replaced the Christoffel function by a "Laguerre" one (14). The last has a similar partial fraction expansion, but allows much more refined estimates.

Our main result is:

Theorem 2.

$$\mathcal{M}\left(\mathbf{P}_k^{(\alpha,\beta)}, \sqrt{1-x^2}\right) \leqslant 11 \left(\frac{(\alpha+\beta+1)^2(2k+\alpha+\beta+1)^2}{4k(k+\alpha+\beta+1)}\right)^{1/3}, \tag{5}$$

provided the parameters of $\mathbf{P}_k^{(\alpha,\beta)}(x)$ *are in the region* $\mathcal{D}$ *defined by*

$$k \geqslant 6, \quad \alpha \geqslant \beta > \tfrac{1}{4}, \quad 16\alpha\beta \geqslant 4\alpha + 4\beta + 1. \tag{6}$$

In particular, (5) *holds for*

$$k \geqslant 6, \quad \alpha \geqslant \beta \geqslant \frac{1+\sqrt{2}}{4}. \tag{7}$$

Despite the attractive numbers, appearance of the region $\mathcal{D}$ is owing rather to our attempts to find a compromise between precision and the amount of calculations required.

We also confirm the conjecture in some limiting cases.

Theorem 3. (i) *There is an absolute constant C such that for a fixed k and sufficiently large* α *and* $\beta = (1-\delta)\alpha$, *with* $\delta = o\left(\alpha^{-1/2}\right)$,

$$\mathcal{M}\left(\mathbf{P}_k^{(\alpha,\beta)}, \sqrt{1-x^2}\right) \leqslant C\sqrt{\alpha}\, k^{-1/6}. \tag{8}$$

Moreover, (8) *holds, yet with C depending on k and* β, *if* $k \to \infty$, $\alpha \to \infty$, $k \ll \alpha$, *and* β *is fixed.*

(ii) *For fixed k and* β,

$$\overline{\lim_{\alpha\to\infty}}\, \alpha^{-1/2}\mathcal{M}\left(\mathbf{P}_k^{(\alpha,\beta)}, \sqrt{1-x^2}\right) < \frac{(4k^2+4\beta k+2\beta+2)^{\beta+\frac{1}{2}}}{\Gamma(\beta+1)} = O(1). \tag{9}$$

In fact, at the cost of some routine calculations, the last theorem may be restated in a non-asymptotic version. Yet, it is worth noticing that the dependence on k and β in (9) is an artefact reflecting rather the lack of sharp bounds for the Laguerre polynomials used in the proof than the real behaviour of $\mathcal{M}$.

Apparently, Theorem 2 and (8) make quite plausible the following refinement of the Erdélyi, Magnus and Nevai conjecture,

Conjecture 1. *The exponent* $\frac{1}{3}$ *in* (5) *can be replaced by* $\frac{1}{6}$.

To simplify formulas in the sequel it will be convenient to use the following set of variables,

$$s = \alpha + \beta + 1, \quad q = \alpha - \beta, \quad r = 2k + \alpha + \beta + 1,$$

and their trigonometric counterparts

$$q = r \sin \omega, \quad s = r \sin \tau, \quad 0 \leqslant \omega, \ \tau < \frac{\pi}{2}.$$

For example the right-hand side in (5) is now written as

$$s^{2/3} r^{2/3} (r^2 - s^2)^{-1/3} = r^{2/3} \tan^{2/3} \tau.$$

Since $\mathbf{P}_k^{(\alpha,\beta)}(x) = (-1)^k \mathbf{P}_k^{(\beta,\alpha)}(-x)$, we may assume $\alpha \geqslant \beta$, therefore everywhere in the sequel $q \geqslant 0$, and $0 \leqslant \omega < \tau$. We also introduce the binary variable $\mathbf{j} \in \{-1, 1\}$. As we will see, in many respects, $q, \ s$ and r are more natural parameters than $k, \ \alpha$ and β. At least they allow to shrink many otherwise awful expressions to a reasonable size.

The idea behind the proof of Theorem 2 is very simple, but requires a substantial amount of calculations which hardly can be done without an appropriate symbolic package. We used Mathematica.

The following pointwise estimate on $M\left(x; \mathbf{P}_k^{(\alpha,\beta)}, \sqrt{1-x^2}\right)$ in the oscillatory region was given in [2].

Theorem 4. *Let* $k \geqslant 1$, *and* $\alpha, \ \beta > -\frac{1}{2}$. *Then for* $-1 < x < 1$,

$$M\left(x; \mathbf{P}_k^{(\alpha,\beta)}, \sqrt{1-x^2}\right) \leqslant \frac{2e}{\pi} \frac{r(r+1)}{(r+1)^2 - 2\alpha^2/(1-x) - 2\beta^2/(1+x)}, \tag{10}$$

provided the denominator $(r+1)^2 - \frac{2\alpha^2}{1-x} - \frac{2\beta^2}{1+x}$ *is positive.*

Theorem 2 is an easy corollary of (10) and the following claim.

Theorem 5. *All local maxima of the function* $M\left(x; \mathbf{P}_k^{(\alpha,\beta)}, \sqrt{1-x^2}\right)$ *are in the interval* (N_{-1}, N_1), *where*

$$N_{\mathbf{j}} = \mathbf{j}\left(\cos(\tau + \mathbf{j}\omega) - \theta_j \left(\frac{\sin^4(\tau + \mathbf{j}\omega)}{2\cos\tau\cos\omega}\right)^{1/3} r^{-2/3}\right),$$

and

$$\theta_{\mathbf{j}} = \begin{cases} \frac{1}{3}, & \mathbf{j} = -1, \\ \frac{3}{10}, & \mathbf{j} = 1, \end{cases} \tag{11}$$

provided the parameters $k, \ \alpha$ *and* β *are in* $\mathcal{D}$.

The bounds given in the last theorem are very precise up to the values of $\theta_{\mathbf{j}}$. This is so because the same expression with two-side bounds on the corresponding constants $\theta_{\mathbf{j}}$ (up to some minor higher order terms) has been obtained for the extreme zeros of Jacobi polynomials [8,10]. Therefore, any improvement of (10) would lead to the corresponding improvement of (5).

Having at hand uniform bounds, we may exploit some limiting relations between orthogonal polynomials in the Askey scheme [3,6] to prove Theorem 3. Uniform bounds for the Hermite polynomials were recently obtained by a method similar to that of this paper [9],

$$C_1 < \mathcal{M}\left(\mathbf{H}_k, 1\right) k^{1/6} < C_2, \tag{12}$$

where $C_1 \approx \frac{1}{2}$, C_2 is a constant, e.g. one can take $C_2 = 10$, for $k \geqslant 2000$, or $C_2 = 10^{63}$, for $k \geqslant 6$. Yet, in the Hermite case we could use very precise inequalities of [4], whereas (10) seems rather poor in the relevant oscillatory region.

The paper is organized as follows. The next two sections deal with the proof of Theorem 5. In Section 2 we reduce the problem to bounding extreme zeros of a six degree polynomial in x being as well a polynomial in parameters α, β and k. The required bounds will be established in a quite technical Section 3. In Section 4 we will deduce Theorem 2 from Theorem 5. A simple proof of Theorem 3 is given in Section 5.

2. Main inequality

Let us stress that in what follows x is restricted to $[-1, 1]$.

The arguments we will present here are rather general and make no use of orthogonality. All that we need is a second order differential equation with a solution being a hyperbolic polynomial, that is a real polynomial with only real zeros, or a uniform limit of such polynomials.

Let $f = f(x) = \mathbf{P}_k^{(\alpha,\beta)}(x)$. Notice that in the sequel we use dash only for derivatives in x. We introduce the logarithmic derivative $t = t(x) = f'(x)/f(x)$. Let also $\boldsymbol{\Omega} = \boldsymbol{\Omega}_k(\alpha, \beta)$ be the set of x corresponding to the local maxima of

$$M\left(x; \mathbf{P}_k^{(\alpha,\beta)}, \sqrt{1-x^2}\right),$$

in x, for given values of α, β and k.

From the equation

$$\frac{d}{dx} M\left(x; \mathbf{P}_k^{(\alpha,\beta)}, \sqrt{1-x^2}\right) = 0,$$

we get

$$t(x) = \frac{sx + q}{2(1-x^2)}, \quad x \in \boldsymbol{\Omega}. \tag{13}$$

Our main tool in bounding the extreme points of Ω will be the following elementary Laguerre inequality,

$$U(p(x)) = \frac{p'^2(x) - p(x)p''(x)}{p^2(x)} = \sum_{i=1}^{k} \frac{1}{(x-x_i)^2} > 0, \tag{14}$$

where $p(x) = c\prod_{i=1}^{k}(x-x_i)$, is a hyperbolic polynomial. Notice that if p is hyperbolic then $p - \lambda p'$ is also hyperbolic for any real λ. Thus, we get

$$U(f - \lambda f') > 0. \tag{15}$$

We have to calculate this expression explicitly. For, we observe that using the standard differential equation for Jacobi polynomials,

$$(1-x^2)f'' = (\alpha - \beta + (\alpha + \beta + 2)x)f' - k(k + \alpha + \beta + 1)f, \tag{16}$$

in a pure algebraic manner one can exclude all the derivatives of f of order greater than one, when such appear. Moreover, $U(f - \lambda f')f^2$ is a quadratic form in f and f'. Hence $U(f - \lambda f')$ can be written as a function of $t = f'/f$. Now applying (13), we obtain that for any $x \in \Omega$ and $\lambda \in \mathbb{R}$,

$$16(1-x^2)^4 U(f - \lambda f') = A_2\lambda^2 - 4(1-x^2)A_1\lambda + 4(1-x^2)^2 A_0 > 0, \tag{17}$$

where

$$\begin{aligned} A_0 &= r^2 - q^2 - s^2 - 2q(s+1)x - (r^2 + 2s)x^2, \\ A_1 &= ((s+3)x + q)A_0 - 2(sx+q)(s+1+qx-x^2), \\ A_2 &= A_0^2 - (sx+q)((s+6)x+q)A_0 - 4(sx+q)^2(s+1+qx-x^2). \end{aligned}$$

To bound Ω we shall find the extreme zeros of the equation

$$F_\lambda(x) = A_2\lambda^2 - 4(1-x^2)A_1\lambda + 4(1-x^2)^2 A_0 = 0. \tag{18}$$

Indeed, the leading term of F_λ is $-4(r^2+2s)x^6$, hence it may be positive only on a bounded interval.

Next, we observe that optimization in λ is straightforward. Namely, viewing $F_\lambda(x)$ as a quadratic in λ, one has to choose these λ for which the discriminant $\Delta = \Delta(x)$, of this quadratic vanishes. This yields, omitting the positive factor $16(1-x^2)^2$,

$$\Delta = A_1^2 - A_0 A_2 \leqslant 0. \tag{19}$$

It is worth noticing that the corresponding optimal values of λ, which we have no need to find, are automatically real. The explicit expression for Δ has a bit surprising form,

$$\Delta = \Big(2(sx+q)(s+1+qx-x^2) - 3xA_0\Big)^2 - A_0^3.$$

Thus, Δ is a polynomial of degree six in x with the positive leading coefficient,

$$\Delta = \Big((r^2+2s)^3 + (3r^2+4s)^2\Big)x^6 + \cdots .$$

All this can be summarized in the following claim.

Lemma 6. *Let y_1 and y_2, $y_1 \leqslant y_2$, be the extreme zeros of the equation*

$$\Delta(x) = 0, \tag{20}$$

then $\Omega \subset (y_1, y_2)$.

So far no essential restrictions are to be imposed on k, α and β, besides $\alpha, \beta > -1$, to force the hyperbolicity. In fact, they come to life just when one wants to find an approximation to the zeros of (20). For small α and β the endpoints of the interval embracing Ω are $\pm\left(1 - O(1/k^2)\right)$. Inequality (15) is simply not strong enough to separate them well from ± 1. The discriminant surface of Δ in abundance of parameters may be very complicated and hardly allows any general treatment. Of course, asymptotics are much easier. For instance, it may be routinely shown that for α and β growing linearly with k, the exact asymptotic value of $\theta_{\mathbf{j}}$ in Eq. (20) is 1. In our case Theorem 5 is an obvious corollary of the Lemma 6 and the following claim.

Lemma 7. *The equation $\Delta = 0$, has precisely two real zeros in $\mathcal{D}$. Moreover,*

$$\Delta(N_{-1}) > 0, \quad \Delta\left(-\frac{qs}{r^2}\right) < 0, \quad \Delta(N_1) > 0, \tag{21}$$

and $N_{-1} < -\frac{qs}{r^2} < N_1$, in $\mathcal{D}$.

The proof of Lemma 7 is quite technically involved and will be given in the next section.

Remark 1. More generally, given a hyperbolic polynomial $p(\lambda) = \sum a_i \lambda^i$, one may consider the expression $U(\sum a_i \lambda^i \ f^{(i)}(x))$, which is positive by the classical Hermite–Poulain theorem (see e.g. [19, p. 14]). Then the bounds on Ω corresponding to the optimal choice of λ are among the roots of the equation $Dis_\lambda U = 0$, where $Dis_\lambda U$ is the discriminant of U in λ, that is the resultant of U and $\frac{dU}{d\lambda}$.

3. Proof of Lemma 7 and Theorem 5

First of all we notice that in terms of s and q the condition $16\alpha\beta > 4\alpha + 4\beta + 1$, appearing in the definition of $\mathcal{D}$ means

$$q^2 \leqslant s^2 - 3s + \tfrac{7}{4}. \tag{22}$$

As it is also assumed $\alpha \geqslant \beta > \frac{1}{4}$, then $s > q \geqslant 0$, and $s \geqslant \frac{3}{2}$. Solving $s^2 - 3s + \frac{7}{4} \geqslant 0$, under these constrains yields

$$s \geqslant \frac{3 + \sqrt{2}}{2}, \quad r \geqslant \frac{27 + \sqrt{2}}{2}. \tag{23}$$

Using this we can define the following change of variables. Let p be a polynomial in each of the variables r, s, q, containing only even powers of q. Denote by $\mathcal{L}(p)$ the polynomial in variables κ, s, h, obtained from p by the substitutions

$$\begin{aligned} &r=2(\kappa+6)+s, \\ &q=\sqrt{s^2(1-h)-3s+\tfrac{7}{4}}, \quad 0\leqslant h\leqslant 1-\frac{12s-7}{4s^2}. \end{aligned} \tag{24}$$

The substitution is real in $\mathcal{D}$ by (22), and $k\geqslant 6$, just means $\kappa\geqslant 0$.

We split the proof of Lemma 7 into several steps. The following claim is the only part of the proof which hardly can be established without a symbolic package, at least if one wants to keep mild restrictions on k, α and β.

Lemma 8. *The equation $\Delta = 0$, has precisely two real zeros in $\mathcal{D}$.*

Proof. To demonstrate that equation $\Delta = 0$ has only two real zeros, we calculate the discriminant $Dis_x(\Delta)$ and show that it does not vanish in $\mathcal{D}$. This implies that the number of real zeros of Δ is the same for any choice of the parameters in $\mathcal{D}$. Choosing $r = 15$, $s = 3$, $q = 0$, that is $k = 6$, $\alpha = \beta = 1$, we obtain the following test equation

$$177755x^6 - 492157x^4 + 454472x^2 - 139968 = 0,$$

which has precisely two roots $x_{1,2} \approx \pm 0.96$. Mathematica gives for the sought discriminant

$$Dis_x(\Delta) = -2^{14}(s^2-q^2)^6(r^2-s^2)^3\left((r^2+2s)^3+(3r^2+4s)^2\right)R_1^3R_2,$$

where

$$R_1 = (r^2+3s+2)^2 - q^2(r^2+4s+3),$$

and R_2 is a polynomial with about 500 terms, of degree 18, 17 and 12 in r, s and q respectively. Moreover, it contains only even powers of q. As $q\leqslant s$, we have

$$R_1 \geqslant (r^2+3s+2)^2 - s^2(r^2+4s+3) = (r^2+4s+2)(r^2-s^2+2s+2) > 0.$$

Thus it is left to show that R_2 does not vanish in $\mathcal{D}$. Applying substitutions (24), we obtain a polynomial $\mathcal{R}(\kappa, s, h) = \mathcal{L}(R_2)$, On expanding $\mathcal{R}$ into monomials it turns out that this polynomial has no negative terms. Moreover, one can check that $\mathcal{R}(0, 0, 0) > 0$, hence $\mathcal{R} > 0$, in $\mathcal{D}$. $\square$

Remark 2. Of course, nothing happens if the resultant may vanish. In this case the equation $\Delta = 0$ may have more than two real roots and one has to choose the extreme ones. For example for $\alpha = \beta = 0$, and $k = 6$ Eq. (20) has six different zeros on $(-1, 1)$.

Remark 3. To avoid the usage of resultants one may try to bound the zeros of the equation $\Delta = 0$ by the Newton–Raphson method. This approach does work in a similar problem for the Laguerre polynomials [12]. Unfortunately, in our case this would require rather tedious calculations (or, at least, we did not find any simple way) to establish the convexity of the involved functions.

To simplify otherwise messy formulas we will define the following functions:

$$B_1(x) = 2\mathbf{j}(sx+q)(2x^2+1),$$
$$B_2(x) = \mathbf{j}\left(3r^2x^3 + 8qsx^2 + (5q^2+5s^2-3r^2)x + 2qs\right),$$
$$C_1(x) = 2x(sx+q),$$
$$C_2(x) = (1-x^2)r^2 - 2qsx - q^2 - s^2.$$

We also put

$$v = \sqrt{r^2-q^2}, \quad u = \sqrt{r^2-s^2}, \quad z_{\mathbf{j}} = sv + \mathbf{j}qu = r^2 \sin(\tau + \mathbf{j}\omega),$$

$$\varepsilon_{\mathbf{j}} = 2^{-1/3}\,\theta_{\mathbf{j}} \left(\frac{z_{\mathbf{j}}}{vu}\right)^{4/3} r^{-2/3}.$$

It is important to stress that v, u and $z_{\mathbf{j}}$ are strictly positive. We will use the above variables in a somewhat mixture way, preferring shorter formulas to a clear separation between algebra and trigonometry.

Now we can rewrite Δ and $N_{\mathbf{j}}$ as follows,

$$\Delta(x) = (B_1(x) + B_2(x))^2 - (C_2(x) - C_1(x))^3, \tag{25}$$

$$N_{\mathbf{j}} = -\frac{qs}{r^2} + \mathbf{j}\,\frac{vu}{r^2}\,(1-\varepsilon_{\mathbf{j}}). \tag{26}$$

First, by $0 \leqslant q < s$, and $r = 2k+s$, we have

$$0 < \frac{z_{\mathbf{j}}}{vu} = \frac{s}{\sqrt{r^2-s^2}} + \mathbf{j}\,\frac{q}{\sqrt{r^2-q^2}} < \frac{2s}{\sqrt{r^2-s^2}} = \frac{s}{\sqrt{k(k+s)}}.$$

Hence, using $\theta_j \leqslant \frac{1}{3}$, we obtain

$$0 < \varepsilon_{\mathbf{j}} < 2^{-1/3}\theta_{\mathbf{j}} \left(\frac{s^2}{k(k+s)(2k+s)}\right)^{2/3} < 2^{-1/3}\theta_{\mathbf{j}} \left(\frac{s}{k+s}\right)^{4/3} k^{-2/3} < \frac{1}{12}. \tag{27}$$

This readily implies the last claim of Theorem 5,

Lemma 9. $N_{-1} < -\frac{qs}{r^2} < N_1$.

The following claim will be useful to simplify calculations.

Lemma 10.

$$B_1(N_{-1}) > 0, \quad B_1(N_1) > 0, \quad C_1(N_{-1}) > 0. \tag{28}$$

Proof. This is the direct corollary of the definitions of $B(N_{\mathbf{j}})$, $C(N_{\mathbf{j}})$ and the following inequalities:

$$sN_{-1} + q < 0, \quad sN_1 + q > 0. \tag{29}$$

Indeed, we have

$$sN_{\mathbf{j}} + q = \mathbf{j}\frac{u(z_{\mathbf{j}} - sv\varepsilon_{\mathbf{j}})}{r^2},$$

hence to prove (29) it is enough to show $\varepsilon_{\mathbf{j}} < \frac{z_{\mathbf{j}}}{sv}$. This is obviously true for $\mathbf{j} = 1$, as $\varepsilon_{\mathbf{j}} < 1$. If $\mathbf{j} = -1$, it also holds, since using $\theta_{\mathbf{j}} < 1$, (22), (23), and $3s - \frac{7}{4} > 2s$, we obtain

$$\left(\frac{z_{-1}}{sv\varepsilon_{-1}}\right)^3 = \frac{2vu^4r^2}{s^3\theta_{-1}^3 z_{-1}} = \frac{2vu^4(sv+qu)}{s^3\theta_{-1}^3(s^2-q^2)} > \frac{2u^6}{s^2(s^2-q^2)} > \frac{u^6}{s^3} > 1. \qquad \square$$

Now we will establish:

Lemma 11. $\Delta(-\frac{qs}{r^2}) < 0$.

Proof. Calculations yield that $\Delta(-\frac{qs}{r^2})$ can be written as $u^4 r^{-12} D(r, s, q)$, where D is a polynomial containing only even powers of q. Applying transformation (24) one gets a polynomial $\mathcal{L}(D)$ without positive terms. $\square$

Now we need some preparations before proving $\Delta(N_{\mathbf{j}}) > 0$. One can easily check that $-1 < N_{-1} < N_1 < 1$. Therefore, we can restrict x to the interval $[-1, 1]$. Calculations give

$$\frac{r^4}{2vuz_{\mathbf{j}}^2} B_2(N_{\mathbf{j}}) = 1 - \frac{3(v^2+u^2)r^2 - z_{\mathbf{j}}^2}{2z_{\mathbf{j}}^2}\,\varepsilon_{\mathbf{j}} + \frac{(9vu - \mathbf{j}qs)vu}{2z_{\mathbf{j}}^2}\,\varepsilon_{\mathbf{j}}^2 - \frac{3v^2u^2}{2z_{\mathbf{j}}^2}\,\varepsilon_{\mathbf{j}}^3, \quad (30)$$

$$C_1(N_{\mathbf{j}}) = u\left(\sin(2\tau + 2\mathbf{j}\omega) - 2\varepsilon_{\mathbf{j}}\cos\omega\sin(2\tau + \mathbf{j}\,\omega) + \varepsilon_{\mathbf{j}}^2\sin 2\tau\cos^2\omega\right). \quad (31)$$

$$\left(\frac{r^4}{2vuz_{\mathbf{j}}^2}\right)^{2/3} C_2(N_{\mathbf{j}}) = \theta_{\mathbf{j}}\left(1 - \frac{\varepsilon_{\mathbf{j}}}{2}\right). \quad (32)$$

In view of (25) it is natural to set

$$H_{\mathbf{j}}^B = \frac{r^4}{2vuz_{\mathbf{j}}^2}\left(B_1(N_{\mathbf{j}}) + B_2(N_{\mathbf{j}})\right),$$

$$H_{\mathbf{j}}^C = \left(\frac{r^4}{2vuz_{\mathbf{j}}^2}\right)^{2/3}\left(C_2(N_{\mathbf{j}}) - C_1(N_{\mathbf{j}})\right) = \left(\frac{r^4}{2vuz_{\mathbf{j}}^2}\right)^{2/3} A_0(N_{\mathbf{j}}).$$

By Lemma 28, $B_1(N_{\mathbf{j}}) > 0$, and we get

$$H_{\mathbf{j}}^B > B_2(N_{\mathbf{j}}) > 1 - \frac{3(v^2+u^2)r^2}{2z_{\mathbf{j}}^2}\,\varepsilon_{\mathbf{j}} + \frac{(9vu - \mathbf{j}qs)vu}{2z_{\mathbf{j}}^2}\,\varepsilon_{\mathbf{j}}^2 - \frac{3v^2u^2}{2z_{\mathbf{j}}^2}\,\varepsilon_{\mathbf{j}}^3. \quad (33)$$

Using the explicit form of $\varepsilon_{\mathbf{j}}$ and simplifying (31), (32) we also obtain

$$H_{\mathbf{j}}^C = \theta_{\mathbf{j}}\left(1 - \frac{r^2\sin(2\tau + 2\mathbf{j}\omega)}{2uv^2\varepsilon_{\mathbf{j}}} - \frac{\varepsilon_{\mathbf{j}}}{2} + \frac{r\sin(2\tau + \mathbf{j}\omega)}{vu}\right). \quad (34)$$

Next, we should provide more convenient estimates for $H_{\mathbf{j}}^B$ and $H_{\mathbf{j}}^C$. We give more accurate bounds for the case $\mathbf{j} = 1$, because, as we see later and is easy to guess, they are more important.

Lemma 12.

$$H_{\mathbf{j}}^B > \begin{cases} 1 - \frac{34}{15}\theta_{-1}, & \mathbf{j} = -1, \\ 1 - \frac{10}{7}\theta_1, & \mathbf{j} = 1. \end{cases} \tag{35}$$

Proof. First we show

$$H_{\mathbf{j}}^B > 1 - \frac{3(v^2+u^2)r^2}{2z_1^2}\,\varepsilon_{\mathbf{j}}. \tag{36}$$

We will use the obvious abbreviation

$$H_{\mathbf{j}}^B = 1 - I_1\varepsilon_{\mathbf{j}} + I_2\varepsilon_{\mathbf{j}}^2 - I_3\varepsilon_{\mathbf{j}}^3.$$

Observe that $I_2\varepsilon_{\mathbf{j}}^2 - I_3\varepsilon_{\mathbf{j}}^3 > 0$, that is $H_{\mathbf{j}}^B < 1 - I_1\varepsilon_{\mathbf{j}}$, thus proving (36) for $\mathbf{j} = -1$. Indeed,

$$\frac{2z_{-1}^2}{vu\varepsilon_{-1}^2}\left(I_2\varepsilon_{-1}^2 - I_3\varepsilon_{-1}^3\right) = 9vu + qs - 3vu\varepsilon_{-1} > 6vu + qs > 0.$$

For $\mathbf{j} = 1$, we obtain by $\varepsilon_1 < \frac{1}{3}$,

$$9vu - qs - 3vu\varepsilon_1 > vu - qs = r^2\cos(\tau + \omega) > -r^2.$$

Thus,

$$I_2\varepsilon_1^2 - I_3\varepsilon_1^3 > -\frac{vur^2}{2z_1^2}\,\varepsilon_1^2. \tag{37}$$

Next, we will show that

$$\frac{\varepsilon_1}{2} + I_2\varepsilon_1^2 - I_3\varepsilon_1^3 > 0, \tag{38}$$

which proves (36) for $\mathbf{j} = 1$. By (37) it is enough check that $z_1^6 - v^3u^3r^6\varepsilon_1^3 > 0$. Since $\theta_{\mathbf{j}} \leqslant 1$, we have

$$\begin{aligned}\frac{2(z_1^6 - v^3u^3r^6\varepsilon_1^3)vuz_1^{-4}}{r^2+\sqrt{2}\,su^2} &= \frac{2vuz_1^2 - r^4}{r^2+\sqrt{2}\,su^2} > \frac{2s^2u^4 - r^4}{r^2+\sqrt{2}\,su^2} \\ &= 4\sqrt{2}\,(k+s)ks - (2k+s)^2 \\ &> 4(k+s)\left(\sqrt{2}\,ks - k - s\right) > 0,\end{aligned}$$

by (23), and (36) follows.

To complete the proof it is left to estimate (36). If $\mathbf{j} = 1$, for the cube of $\frac{3(v^2+u^2)r^2}{2z_1^2\theta_1}\varepsilon_1$ we obtain

$$\frac{27r^4(v^2+u^2)^3}{16v^4u^4(sv+qu)^2} \leqslant \frac{27r^4(v^2+u^2)^3}{16s^2u^4v^6} \leqslant \frac{27r^4}{2s^2(r^2-s^2)^2} = \frac{27(2k+s)^4}{32k^2s^2(k+s)^2} < \left(\frac{10}{7}\right)^3.$$

Similarly, for $\mathbf{j} = -1$, using (22) we get

$$\frac{27r^4(v^2+u^2)^3}{16v^4u^4(sv-qu)^2} \leqslant \frac{27v^2(sv+qu)^2}{2u^4(s^2-q^2)^2} < \frac{54s^2v^4}{u^4(s^2-q^2)^2} \leqslant \frac{27(16k^2+16ks+12s-7)s^2}{8(k+s)^2(12s-7)^2k^2} < \left(\frac{34}{15}\right)^3,$$

giving the required estimate. □

Lemma 13.

$$H_{\mathbf{j}}^C \leqslant \begin{cases} \theta_{-1}, & \mathbf{j} = -1, \\ \dfrac{3^{-1/3}}{2} + \dfrac{23}{24}\theta_1 - \dfrac{3^{-2/3}}{4}\theta_1^2, & \mathbf{j} = 1. \end{cases} \tag{39}$$

Proof. If $\mathbf{j} = -1$, then $H_{-1}^C \leqslant C_2(N_1)$, by Lemma 28, which yields the result.

For $\mathbf{j} = 1$, we proceed in a straightforward manner, evaluating honestly all required extrema.

$$3\left(u^4v^7z_1^4r^{-2}\right)^{1/3} \frac{\partial}{\partial\omega} H_1^C = J_1 - J_2 + J_3,$$

where

$$J_1^3 = 2v^2u^5(vr^2+2sz_1)^3,$$
$$J_2^3 = 4r^2\theta_1^6u^3z_1^5,$$
$$J_3^3 = 3r^4\theta_1^3uvz_1^4\cos^3 2\tau.$$

The derivative is positive since, using the extreme values of k and s, we find

$$(J_2/J_1)^3 = \frac{2\theta_1^6r^2z_1^5}{(r^2v+2sz_1)^3u^2v^2} \leqslant \frac{64\theta_1^6s^5}{(r^2-s^2)r^4} < \left(\frac{\theta_1^2}{13}\right)^3,$$

$$(J_3/J_1)^3 = \frac{27\theta_1^3r^4z_1^4\cos^3 2\tau}{2(2sz_1+vr^2)^3vu^4} \leqslant \frac{216\theta_1^3s^4}{(r^2-s^2)^2r^2} < \lim_{s\to\infty} \frac{27\theta_1^3s^4}{2(2k^3+3sk^2+ks^2)^2} \leqslant \frac{3\theta_1^3}{8}.$$

and

$$\frac{3^{1/3}\theta_1}{2} + \frac{\theta_1^2}{13} < 1.$$

Therefore, the maximum of H_1^C is attained for $\omega = \tau$, i.e.

$$H_1^C \leqslant \theta_1 - \frac{2u^2 - r^2}{s^{1/3} r^{4/3} u^{2/3}} - \frac{s^{4/3}}{u^{4/3} r^{2/3}} \theta_1^2 + \frac{(4u^2 - r^2)s}{u^2 r^2} \theta_1 := \mathcal{H}. \tag{40}$$

Similarly,

$$\frac{u^3}{s^{4/3} r^{8/3}} \frac{\partial}{\partial \tau} \mathcal{H} = L_1 - L_2 - L_3,$$

where

$$L_1^3 = \tfrac{1}{8}(9 - 2\cos 3\tau - 3\cos 4\tau)^3 u^4 r^{-2} \geqslant 8u^4 r^{-2},$$
$$L_2^3 = 64 s^5 u^2 r^{-6} \theta_1^6,$$
$$L_3^3 = 27(5\cos 2\tau + \cos 4\tau)^3 s^4 r^{-4} \theta_1^3 \leqslant 162 s^4 r^{-4} \theta_1^3,$$

$$(L_2/L_1)^3 \leqslant \frac{8s^5\theta_1^6}{u^4 r^4} \leqslant \frac{2\theta_1^6}{k} \lim_{s\to\infty} \frac{s^5}{(k+s)(2k+s)^4} \leqslant \frac{\theta_1^6}{3},$$

$$(L_3/L_1)^3 \leqslant \frac{81 s^4 \theta_1^3}{4u^4 r^2} < \frac{81\theta_1^3}{64k^2} \lim_{s\to\infty} \frac{s^4}{(k+s)^2(2k+s)^2} \leqslant \frac{9\theta_1^3}{256}.$$

Checking that

$$3^{-1/3}\theta_1^2 + 3^{2/3} 2^{-8/3} \theta_1 < 1,$$

for $\theta_1 < \frac{80}{81}$, we conclude that H_1^c attains the maximum at the largest possible value of τ. Finally, setting

$$\mathcal{H} = \theta_1 - \mathcal{H}_1 - \mathcal{H}_2 \theta_1^2 + \mathcal{H}_3 \theta_1,$$

where the terms are listed in the same order as in (40), and taking the limit we obtain

$$\mathcal{H}_1^3 < \lim_{s\to\infty} \frac{(r^2 - 2s^2)^3}{(r^2 - s^2) s r^4} = -\frac{1}{4k},$$

$$\mathcal{H}_2^3 < \lim_{s\to\infty} \frac{s^4 \theta_1^6}{u^4 r^2} = \frac{\theta_1^6}{16k^2},$$

$$\mathcal{H}_3 < \lim_{s\to\infty} \frac{(3r^2 - 4s^2)\theta_1}{(r^2 - s^2) r^2} = -\frac{\theta_1}{4k},$$

where we substituted $r = 2k + s$ to find the limits. Thus,

$$\begin{aligned} H_1^C &< \max_{k\geqslant 6} \left\{ (4k)^{-1/3} + \left(1 - \frac{1}{4k}\right)\theta_1 - (4k)^{-2/3}\theta_1^2 \right\} \\ &= \frac{3^{-1/3}}{2} + \frac{23}{24}\theta_1 - \frac{3^{-2/3}}{4}\theta_1^2, \end{aligned}$$

where the maximum is attained for $k = 6$, provided $\theta_1 < \frac{31}{32}$. $\square$

Now it is just a matter of straightforward calculations to complete the proof of Lemma 7 and thus, of Theorem 5. Substituting the bounds of (35) and (39), and the values of $\theta_{\mathbf{j}}$ one checks that indeed $B(N_{\mathbf{j}})^2 - C(N_{\mathbf{j}})^2 > 0$, and so $D(N_{\mathbf{j}}) > 0$.

4. Proof of Theorem 2

In the following two lemmas we collect some technical claims we use in the proof of Theorem 2.

Lemma 14.

$$-\frac{q}{s} \in (N_{-1}, N_1)\,.$$

Proof. It follows from the explicit formula

$$\Delta\left(-\frac{q}{s}\right) = -(s^2-q^2)^2(r^2-s^2)^2(u^2(u^2-v^2)-9q^2)s^{-6} < 0,$$

hence $N_{-1} < \frac{q}{s} < N_1$. $\square$

Lemma 15.

$$1 - N_{\mathbf{j}}^2 \leqslant \rho_{\mathbf{j}} z_{\mathbf{j}}^2 r^{-4} \tag{41}$$

where

$$\rho_{\mathbf{j}} = \begin{cases} \frac{3}{2}, & \mathbf{j} = -1, \\ \frac{9}{7}, & \mathbf{j} = 1. \end{cases}$$

Proof. We calculate

$$\begin{aligned} 1 - N_{\mathbf{j}}^2 &= \frac{z_{\mathbf{j}}^2}{r^4}\left(1 + \frac{2vur^2\cos(\tau+\mathbf{j}\omega)}{z_{\mathbf{j}}^2}\varepsilon_{\mathbf{j}} - \frac{v^2u^2}{z_{\mathbf{j}}^2}\varepsilon_{\mathbf{j}}^2\right) \\ &< \frac{z_{\mathbf{j}}^2}{r^4}\left(1 + \frac{2vur^2\cos(\tau+\mathbf{j}\omega)}{z_{\mathbf{j}}^2}\varepsilon_{\mathbf{j}} -\right) := \frac{z_{\mathbf{j}}^2}{r^4}\big(1 + S_{\mathbf{j}}(\tau,\omega)\big)\,, \end{aligned} \tag{42}$$

$$\frac{\partial}{\partial\omega} S_{\mathbf{j}}^3 = -4\mathbf{j}\,\frac{(2vz_{\mathbf{j}} + sr^2)r^6}{uv^2z_{\mathbf{j}}^4}\,\theta_{\mathbf{j}}^3. \tag{43}$$

Hence

$$S_1(\tau,\omega) < S_1(\tau,0) = \theta_1\left(\frac{2u}{sr}\right)^{2/3} < \frac{2}{7}.$$

For $\mathbf{j} = -1$ we have to take the maximal possible value of ω, i.e. q given by (22), which yields

$$S_{-1}(\tau, \omega) < \theta_{-1}\left(\frac{4s}{s^2 - q^2}\right)^{2/3} \leqslant 4\theta_{-1}\left(\frac{2s}{12s - 7}\right)^{2/3} < \frac{1}{2}. \qquad \square$$

Let now

$$\mathcal{F}(x) = (r+1)^2 - \frac{2\alpha^2}{1-x} - \frac{2\beta^2}{1+x} = (r+1)^2 - \frac{(s+q-1)^2}{2(1-x)} - \frac{(s-q-1)^2}{2(1+x)}$$

be the denominator of (10). Then to estimate the right-hand side of (10) all we have to find is

$$\min_{N_{-1} \leqslant x \leqslant N_1} \mathcal{F}(x).$$

We will put

$$\mathcal{F}(x) = \mathcal{F}_1(x) + \frac{2s + 2qx - 1}{1 - x^2} + 2r + 1 > \mathcal{F}_1(x),$$

with

$$\mathcal{F}_1 = r^2 - \frac{(s+q)^2}{2(1-x)} - \frac{(s-q)^2}{2(1+x)}. \tag{44}$$

As

$$\frac{\partial}{\partial x}\mathcal{F}_1 = -\frac{2(s+qx)(q+sx)}{(1-x^2)^2},$$

then $\mathcal{F}_1$ has only the local maximum at $x = -q/s$ on $[-1, 1]$. Thus, by Lemma (14), we conclude that

$$\min_{N_{-1} \leqslant x \leqslant N_1} \mathcal{F}_1(x) = \min\{\mathcal{F}_1(N_{-1}), \mathcal{F}_1(N_1)\}.$$

We find with $\varepsilon_{\mathbf{j}} < \frac{1}{12}$,

$$\mathcal{F}_1(N_{\mathbf{j}}) = \frac{(2 - \varepsilon_{\mathbf{j}})u^2 v^2}{(1 - N_{\mathbf{j}}^2)r^2}\varepsilon_{\mathbf{j}} > \frac{23u^2v^2}{12(1 - N_{\mathbf{j}}^2)r^2}\,\varepsilon_{\mathbf{j}}. \tag{45}$$

Substituting this into (10) and taking into account (23) yields

$$\begin{aligned} \mathcal{N} &< \frac{24e}{23\pi}\,\frac{r+1}{r}\,\frac{r^4}{u^2v^2}\,\max_{\mathbf{j}}\left\{\frac{1 - N_{\mathbf{j}}^2}{\varepsilon_{\mathbf{j}}}\right\} \\ &< \frac{78}{77}\max_{\mathbf{j}}\left\{\left(1 - N_{\mathbf{j}}^2\right)\left(\frac{2r^7}{uvz_{\mathbf{j}}^2}\right)^{2/3}\theta_{\mathbf{j}}^{-1}\right\}. \end{aligned} \tag{46}$$

In the last expression we want to get rid of the dependance on q. To that end we apply (41) yielding

$$M\left(x; \mathbf{P}_k^{(\alpha,\beta)}, \sqrt{1-x^2}\right) < \frac{78}{77} \max_{\mathbf{j}} \left\{ \theta_{\mathbf{j}}^{-1} \max_{q} T_{\mathbf{j}} \right\},$$

where

$$T_{\mathbf{j}} = \left(\frac{2rz_{\mathbf{j}}}{vu}\right)^{2/3}.$$

As one can check $T_{\mathbf{j}}$ is an increasing function in q for $\mathbf{j} = 1$, and a decreasing one otherwise. Finding the corresponding extrema we obtain

$$T_{-1}^3 < \frac{4s^2r^2}{r^2 - s^2}, \quad T_1^3 = \frac{16s^2r^2}{r^2 - s^2}.$$

Substituting these into (46) yields the required constant 11, with the maximum attained for $\mathbf{j} = 1$. This completes the proof of Theorem 2.

Notice that for $\mathbf{j} = -1$ one obtains rather large $7.2\ldots$, instead of 11 for the constant, mainly due to our careless estimates in this case. Another reason is that for $j = 1$ the extremum attains in the ultraspherical case $\alpha = \beta$ versus the largest possible value of $\alpha - \beta$ for $j = -1$. But the last case is again almost the ultraspherical one because of the well known relations

$$\frac{(\lambda)_k}{(1/2)_k} P_k^{(\lambda-\frac{1}{2},-\frac{1}{2})}(2x^2-1) = C_{2k}^{(\lambda)}(x),$$
$$x\frac{(\lambda)_{k+1}}{(1/2)_{k+1}} P_k^{(\lambda-\frac{1}{2},\frac{1}{2})}(2x^2-1) = C_{2k+1}^{(\lambda)}(x),$$

where $C_k^{(\lambda)}(x)$ are Gegenbauer polynomials.

5. Asymptotics

The main aim of this section to prove Theorem 3. We split the proof into three lemmas which exploit some known limiting relation between Jacobi, Hermite and Laguerre polynomials. All norms appearing in the sequel are the standard weighted L_2 norms of the corresponding polynomials [20]. We will use the sign $\approx$ to indicate an asymptotic equality with a multiplicative $1 + o(1)$ constant. Let us also remind that regular letters indicate the standard normalization.

Lemma 16. *Let k be fixed, α and $\beta = (1-\delta)\alpha$, sufficiently large, with $\delta = o\left(\alpha^{-1/2}\right)$. Then*

$$\mathcal{M}\left(\mathbf{P}_k^{(\alpha,\beta)}, \sqrt{1-x^2}\right) \leqslant C\sqrt{\alpha}\, k^{-1/6},$$

where the constant $C \in [C_1, C_2]$, with C_1, C_2 defined in (12).

Proof. As we want to deviate from the pure ultraspherical case, we apply a recently established limiting relation between Jacobi and Hermite polynomials [3],

$$\lim_{s\to\infty} P_k^{(\alpha,\beta)}\left(\frac{\sqrt{2s-2}x+q}{s+1}\right)s^{-k/2}=\frac{H_k(x)}{2^{3k/2}k!},\quad q/s\to 0. \tag{47}$$

By $\beta=(1-\delta)\alpha$, with $\delta=o\left(\alpha^{-1/2}\right)$, we have for sufficiently large α,

$$y=\frac{\sqrt{2s-2}x+q}{s+1}\approx\frac{x}{\sqrt{\alpha}},$$

and

$$(1-y)^{\alpha+1/2}(1+y)^{\beta+1/2}\approx e^{-x^2}.$$

Then, by (47), we obtain,

$$\begin{aligned}&\frac{1}{2^{3k}k!^2}\max_x\left\{(1-y)^{\alpha+1/2}(1+y)^{\beta+1/2}\,(H_k(x))^2\right\}\\&\approx\frac{1}{4^k k!}\max_x\left\{(\mathbf{H}_k(x))^2\,e^{-x^2}\right\}\\&\approx\mathcal{M}\left(\mathbf{P}_k^{(\alpha,\beta)}(x),\sqrt{1-x^2}\right)\left\|P_k^{(\alpha,\beta)}\right\|^2 s^{-k}.\end{aligned}$$

Applying (12) and Stirling's approximation for the norm of the Jacobi polynomials, we obtain

$$\mathcal{M}\left(\mathbf{P}_k^{(\alpha,\beta)}(x),\sqrt{1-x^2}\right)\approx C\frac{k^{-1/6}s^k}{2^{2k}k!\left\|P_k^{(\alpha,\beta)}\right\|^2}<C\sqrt{\frac{\alpha}{\pi}}\,k^{-1/6}, \tag{48}$$

where we can take $C_1<C<C_2$, with C_1, C_2 defined in (12). This completes the proof. □

Remark 4. The assumption $\beta=\alpha-o\left(\alpha^{-1/2}\right)$, is made to avoid well-known messy technicalities in approximations of binomial coefficients. It is interesting to see what constant would be obtained in (48) if we were allowed to substitute for C its limiting value with $k\to\infty$. The asymptotic for Hermite case is well known [20] and yields C about 0.5.

Remark 5. The meaning of limiting relations in our case is that the standard differential equation (16) for Jacobi polynomials is just a perturbation of the corresponding Hermite one. As, possessing uniform bounds, one can readily estimate the difference between the solutions, a quantitative version of (47) is available. This enables one to allow k slowly growing with α. Whereas to obtain whatever bound of this type is an easy task, to provide a good one seems a difficult problem and we are not aware of any result in this direction. Apparently less formal reason for limiting relations is that in a certain range of parameters the zeros of one polynomial interlace with these of another. But, as far as we know, this has never been properly established.

Lemma 17. *For large α and k, $k \ll \alpha$, and β fixed,*

$$\mathcal{M}\left(\mathbf{P}_k^{(\alpha,\beta)}, \sqrt{1-x^2}\right) \sim \sqrt{\alpha}k^{-1/6}.$$

Proof. Now we consider a limiting relations between Jacobi and Laguerre polynomials $L_k^{(\alpha)}$.

$$\lim_{\alpha\to\infty} P_k^{(\alpha,\beta)}\left(\frac{2x}{\alpha}-1\right) = (-1)^k L_k^{(\beta)}(x). \tag{49}$$

As in the proof of Lemma 16 we obtain the following relation between two (in fact, unknown) maxima.

$$\mathcal{M}\left(\mathbf{P}_k^{(\alpha,\beta)}, \sqrt{1-x^2}\right) \left\| P_k^{(\alpha,\beta)} \right\|^2 \approx 2^{\alpha+\beta+1}\alpha^{\beta-\frac{1}{2}} \mathcal{M}\left(\mathbf{L}_k^{(\beta)}, \sqrt{x}\right) \left\| L_k^{(\beta)} \right\|^2.$$

On getting rid of gamma functions, we obtain for large α and fixed β and k,

$$\sqrt{\alpha}\,\mathcal{M}\left(\mathbf{L}_k^{(\beta)}, \sqrt{x}\right) \approx \mathcal{M}\left(\mathbf{P}_k^{(\alpha,\beta)}, \sqrt{1-x^2}\right). \tag{50}$$

We allow growing k, which can be routinely justified (see Remark 5), and take a classical Szegö result [20, Theorem 8.91.1], saying in the orthonormal case with $k \to \infty$, and β fixed,

$$\mathcal{M}\left(\mathbf{L}_k^{(\beta)}, \sqrt{x}\right) \sim k^{-1/6}.$$

This completes the proof of the lemma as well as Theorem 3. □

Very few explicit upper bounds are known for Laguerre polynomials, best of which is probably the classical one due to Szegö (see e.g. [1]; a sharper inequality given in [18] is not explicit and hardly could be used),

$$\left(\mathbf{L}_k^{(\alpha)}(x)\right)^2 \leqslant \frac{e^x}{\Gamma(\alpha+1)}, \quad x, \alpha \geqslant 0. \tag{51}$$

Unfortunately, because of the extra $\sqrt{x}$ we have to restrict x first to the oscillatory region which gives a weak estimate. On the other hand, if our Conjecture 1 is true, or any other sharper bounds would be found, this automatically yields an estimate of the Laguerre polynomials.

Lemma 18.

$$\mathcal{M}\left(\mathbf{L}_k^{(\alpha)}, \sqrt{x}\right) < \frac{(4k^2+4\alpha k+2\alpha+2)^{\alpha+\frac{1}{2}}}{\Gamma(\alpha+1)}. \tag{52}$$

Proof. We just repeat a simple part of arguments of Section 2 for the Laguerre case in order to make a full use of Szegö' inequality (51). We need an upper bound on the location of the relative maxima of $e^{-x}x^{\alpha+\frac{1}{2}}$. In the absence of an analogue of (10), it makes no sense to seek estimates as precise as (5) (in fact, this time it is much easier, as we have one parameter less, see [12]). We assume $x \geqslant 0$. Applying Laguerre inequality (14) (no λ this time) and excluding higher derivatives by the differential equation

$$xy'' + (\alpha + 1 - x)y' + k(k + \alpha)y = 0, \quad y = L_k^{(\alpha)},$$

one obtains

$$y^{-2}\left(k(k+\alpha)y^2 + (\alpha + 1 - x)yy' + xy'^2\right) > 0,$$

or, introducing the logarithmic derivative $t(x) = y'/y$,

$$k(k + \alpha) + (\alpha + 1 - x)t + xt^2 > 0. \tag{53}$$

From the condition

$$\frac{d}{dx}\left(e^{-x}x^{\alpha+1/2}y^2\right) = 0,$$

we get

$$t(x) = \frac{2x - 2\alpha - 1}{4x}.$$

Substituting this into (14) yields

$$-4x^2 + 8(2k^2 + 2\alpha k + \alpha + 1)x - (2\alpha + 1)(2\alpha + 3) > 0.$$

The greater root of this quadratic is less than $4k^2 + 4ak + 2a + 2$, and this is the sought, even, as on can check, asymptotically sharp bound on the location of the last maximum. Now the result follows from (51). $\square$

Finally, the inequality (9) of Theorem 8 is an immediate corollary of (50) and (52).

Remark 6. Using in a similar way a limiting relation in the Askey scheme between two polynomials p_k and q_k [3,6,16], and assuming that $\mathcal{M}(\mathbf{p}_k, \phi)$ is known, one can readily deduce some information on $\mathcal{M}(\mathbf{q}_k, \phi^*)$. Moreover, ϕ^* is uniquely defined by the limiting relations and the choice of ϕ for $\mathbf{p}_k$, as a result of our convention to keep the same auxiliary function for the entire family. Thus, starting with, e.g. Jacobi polynomials, one can apply exact bounds, asymptotics or inequalities to obtain something (and maybe guess the true order), say, for Wilson, Hahn or continuous Hahn polynomials.

More accurate calculations in Lemma 16, without the restriction $\beta = \alpha - o\left(\alpha^{-1/2}\right)$, as well as some numerical evidences, suggest the following conjecture, which, if true, would reduce estimates of $\mathcal{M}\left(\mathbf{P}_k^{(\alpha,\beta)}, \sqrt{1-x^2}\right)$ to much easier ultraspherical case.

Conjecture 2. *$\mathcal{M}\left(\mathbf{P}_k^{(\alpha,\beta)}(x), \sqrt{1-x^2}\right)$ is an increasing function in β for $\alpha \geqslant \beta > -\frac{1}{2}$.*

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References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1964.
[2] T. Erdélyi, A.P. Magnus, P. Nevai, Generalized Jacobi weights, Christoffel functions, and Jacobi polynomials, SIAM J. Math. Anal. 25 (1994) 602–614.
[3] C. Ferreira, J.L. Lopez, E. Mainar, Asymptotic relations in the Askey scheme for hypergeometric orthogonal polynomials, Adv. Appl. Math. 31 (2003) 61–85.
[4] W.H. Foster, I. Krasikov, Explicit bounds for Hermite polynomials in the oscillatory region, LMS J. Comput. Math. 3 (2000) 307–314.
[5] C. Giordano, A. Laforgia, On the Bernstein-type inequalities for ultraspherical polynomials, J. Comput. Appl. Math. 153 (2003) 243–248.
[6] R. Koekoek, R.F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Reports of the Faculty of Technical Mathematics and Informatics No. 98-17, Delft, 1998 (math.CA/9602214 at arXiv.org).
[7] I. Krasikov, Nonnegative quadratic forms and bounds on orthogonal polynomials, J. Approx. Theory 111 (2001) 31–49.
[8] I. Krasikov, On zeros of polynomials and allied functions satisfying second order differential equation, East J. Approx. 9 (2003) 51–65.
[9] I. Krasikov, New bounds on the Hermite polynomials, East J. Approx. 10 (2004) 1–8.
[10] I. Krasikov, On extreme zeros of classical orthogonal polynomials, J. Comp. Appl. Math., to appear.
[12] I. Krasikov, Inequalities for Laguerre polynomials, East J. Approx., to appear.
[13] A.L. Levin, D.S. Lubinsky, Bounds for orthogonal polynomials for exponential weights, J. Comput. Appl. Math. 99 (1998) 475–490.
[14] A.L. Levin, D.S. Lubinsky, Asymptotics associated with exponential weights, Intern. Math. Research Notices 12 (1999) 673–683.
[15] E. Levin, D.S. Lubinsky, Orthogonal Polynomials with Exponential Weights, Canadian Mathematical Society Books in Maths, vol. 4, Springer, New York, 2001.
[16] J.L. Lopez, N.M. Temme, The Askey scheme for hypergeometric orthogonal polynomials viewed from asymptotic analysis, J. Comput. Appl. Math. 133 (2001) 623–633.
[17] A. Máté, P. Nevai, V. Totik, Oscillatory behavior of orthogonal polynomials, Proc. Amer. Math. Soc. 96 (1986) 261–268.
[18] M. Michalska, J. Szynal, A new bound for the Laguerre polynomials, J. Comput. Appl. Math. 133 (2001) 489–493.
[19] N. Obreschkoff, Verteilung und Berechnung der Nullstellen reeller Polynome, VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.
[20] G. Szegö, Orthogonal Polynomials, American Mathematical Society Colloquium Publications, vol. 23, Providence, RI, 1975.